

Nonlinear Connections and Exact Solutions in Einstein and Extra Dimension Gravity

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Abstract

We outline a new geometric method of constructing exact solutions of gravitational field equations parametrized by generic off-diagonal metrics, anholonomic frames and possessing, in general, nontrivial torsion and nonmetricity. The formalism of nonlinear connections is elaborated for (pseudo) Riemannian and Einstein–Cartan–Weyl spaces.

1 Introduction: Nonlinear Connections

We consider a $(n+m)$ -dimensional manifold V^{n+m} , provided with general metric and linear connection structure and of necessary smooth class. It is supposed that in any point $u \in V^{n+m}$ there is a local splitting into n - and m -dimensional subspaces, $V_u^{n+m} = V_u^n \oplus V_u^m$. The local/abstract coordinates are denoted $u = (x, y)$, or $u^\alpha = (x^i, y^a)$, where $i, j, k, \dots = 1, 2, \dots, n$ and $a, b, c, \dots = n+1, n+2, \dots, n+m$. The metric is parametrized in the form

$$\mathbf{g} = g_{ij}(u)\mathbf{e}^i \otimes \mathbf{e}^j + h_{ab}(u)\mathbf{e}^a \otimes \mathbf{e}^b \quad (1)$$

where

$$\vartheta^\mu = [\vartheta^i = dx^i, \vartheta^a = dy^a + N_i^a(u)dx^i] \quad (2)$$

is the dual frame to

$$\mathbf{e}_\nu = [e_i = \frac{\partial}{\partial x^i} - N_i^a(u)\frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a}]. \quad (3)$$

Let us denote by $\pi^\top : TV^{n+m} \rightarrow TV^n$ the differential of a map $\pi : V^{n+m} \rightarrow V^n$ defined by fiber preserving morphisms of the tangent bundles TV^{n+m} and

TV^n . The kernel of π^\top is just the vertical subspace vV^{n+m} with a related inclusion mapping $i : vV^{n+m} \rightarrow TV^{n+m}$.

Definition 1 A nonlinear connection (N -connection) \mathbf{N} on space V^{n+m} is defined by the splitting on the left of an exact sequence

$$0 \rightarrow vV^{n+m} \rightarrow TV^{n+m} \rightarrow TV^{n+m}/vV^{n+m} \rightarrow 0,$$

i. e. by a morphism of submanifolds $\mathbf{N} : TV^{n+m} \rightarrow vV^{n+m}$ such that $\mathbf{N} \circ i$ is the unity in vV^{n+m} .

Equivalently, a N -connection is defined by a Whitney sum of horizontal (h) subspace (hV^{n+m}) and vertical (v) subspaces,

$$TV^{n+m} = hV^{n+m} \oplus vV^{n+m}. \quad (4)$$

A space provided with N -connection structure will be denoted V_N^{n+m} . We shall use boldfaced indices for the geometric objects adapted to N -connections. The well known class of linear connections consists a particular subclass with the coefficients being linear on y^a , i. e. $N_i^a(u) = \Gamma_{bj}^a(x)y^b$.

To any sets $N_i^a(u)$, we can associate certain anholonomic frames (2) and (3), with associated N -connection structure, satisfying the anholonomy relations

$$[\vartheta_\alpha, \vartheta_\beta] = \vartheta_\alpha \vartheta_\beta - \vartheta_\beta \vartheta_\alpha = W_{\alpha\beta}^\gamma \vartheta_\gamma$$

with (antisymmetric) nontrivial anholonomy coefficients $W_{ia}^b = \partial_a N_i^b$ and $W_{ji}^a = \Omega_{ij}^a$, where $\Omega_{ij}^a = e_{[i} N_{j]}^a$ are the coefficients of the N -connection curvature.

Essentially, the method to be presented in this work is based on the notion of N -connection and considers a Whitney-like splitting of the tangent bundle to a manifold into a horizontal (see discussion and bibliography in Refs. [1, 2, 3]). Here we emphasize that the geometrical aspects of the N -connection formalism has been studied since the first papers of E. Cartan [4] and A. Kawaguchi [5, 6] (who used it in component form for Finsler geometry), then one should be mentioned the so called Ehresmann connection [7]) and the work of W. Barthel [8] where the global definition of N -connection was given. The monograph [9] consider the N -connection formalism elaborated and applied to geometry of generalized Finsler–Lagrange and Cartan–Hamilton spaces. There is a set of contributions by Spanish authors, see, for instance, [10, 11, 12].

We considered N -connections for Clifford and spinor structures [13, 14], on superbundles and (super) string theory [15] as well in noncommutative geometry and gravity [16]. The idea to apply the N -connections formalism as a new geometric method of constructins exact solutions in gravity theories was suggested in Refs. [17, 18] and developed and applied in a number of works, see for instance, Ref. [19, 20, 21]). This contribution outlines the author's and co-authors' results.

2 N-distinguished Torsions and Curvatures

The geometric constructions can be adapted to the N-connection structure:

Definition 2 A distinguished connection (d-connection) $\mathbf{D} = \{\Gamma_{\beta\gamma}^\alpha\}$ on V_N^{n+m} is a linear connection conserving under parallelism the Whitney sum (4).

Any d-connection \mathbf{D} is represented by irreducible h- v-components $\Gamma_{\beta\gamma}^\alpha = (L_{jk}^i, \tilde{L}_{bk}^a, C_{jc}^i, \tilde{C}_{bc}^a)$ stated with respect to N-elongated frames (2) and (3). This defines a N-adapted splitting into h- and v-covariant derivatives, $\mathbf{D} = D^{[h]} + D^{[v]}$, where $D^{[h]} = (L, \tilde{L})$ and $D^{[v]} = (C, \tilde{C})$. A d-tensor (distinguished tensor, for instance, a d-metric like (1)) formalism and d-covariant differential and integral calculus can be elaborated [1] for spaces provided with general N-connection, d-connection and d-metric structure and nontrivial nonmetricity

$$\mathcal{Q}_{\alpha\beta} \stackrel{\circ}{=} -\mathbf{D}\mathbf{g}_{\alpha\beta}.$$

The simplest way is to use N-adapted differential forms like $\Gamma_\beta^\alpha = \Gamma_{\beta\gamma}^\alpha \vartheta^\gamma$ with the coefficients defined with respect to (2) and (3).

Theorem 3 The torsion $T^\alpha \stackrel{\circ}{=} \mathbf{D}\vartheta^\alpha = d\vartheta^\alpha + \Gamma_\beta^\alpha \wedge \vartheta^\beta$ of a d-connection has the irreducible h- v- components (d-torsions),

$$\begin{aligned} T_{jk}^i &= L_{[jk]}^i, \quad T_{ja}^i = -T_{aj}^i = C_{ja}^i, \quad T_{ji}^a = \Omega_{ji}^a, \\ T_{bi}^a &= T_{ib}^a = \frac{\partial N_i^a}{\partial y^b} - L_{bi}^a, \quad T_{bc}^a = C_{[bc]}^a. \end{aligned} \quad (5)$$

Proof. By a straightforward calculation we can verify the formulas.

The Levi-Civita linear connection $\nabla = \{\nabla \Gamma_{\beta\gamma}^\alpha\}$, with vanishing torsion and nonmetricity, is not adapted to the global splitting (4). One holds:

Proposition 4 There is a preferred, canonical d-connection structure, $\widehat{\mathbf{D}}$, on V_N^{n+m} , constructed only from the metric coefficients $[g_{ij}, h_{ab}, N_i^a]$ and satisfying the conditions $\widehat{\mathbf{Q}}_{\alpha\beta} = 0$ and $\widehat{T}_{jk}^i = 0$ and $\widehat{T}_{bc}^a = 0$.

Proof. By straightforward calculations with respect to the N-adapted bases (2) and (3), we can verify that the connection

$$\widehat{\Gamma}_{\beta\gamma}^\alpha = \nabla \Gamma_{\beta\gamma}^\alpha + \widehat{\mathbf{P}}_{\beta\gamma}^\alpha \quad (6)$$

with the deformation d-tensor

$$\widehat{\mathbf{P}}_{\beta\gamma}^\alpha = (P_{jk}^i = 0, P_{bk}^a = \frac{\partial N_k^a}{\partial y^b}, P_{jc}^i = -\frac{1}{2}g^{ik}\Omega_{kj}^a h_{ca}, P_{bc}^a = 0)$$

satisfies the conditions of this Proposition. It should be noted that, in general, the components \widehat{T}_{ja}^i , \widehat{T}_{ji}^a and \widehat{T}_{bi}^a are not zero. This is an anholonomic frame (or, equivalently, off-diagonal metric) frame effect.

The torsion of the connection (6) is denoted $\widehat{\mathbf{T}}_{\beta\gamma}^\alpha$. In a similar form we can prove:

Theorem 5 *The curvature $\mathcal{R}_\beta^\alpha \doteq \mathbf{D}\Gamma_\beta^\alpha = d\Gamma_\beta^\alpha - \Gamma_\beta^\gamma \wedge \Gamma_\gamma^\alpha$ of a d-connection Γ_γ^α has the irreducible h- v- components (d-curvatures),*

$$\begin{aligned} R^i_{hjk} &= e_k L^i_{hj} - e_j L^i_{hk} + L^m_{hj} L^i_{mk} - L^m_{hk} L^i_{mj} - C^i_{ha} \Omega^a_{kj}, \\ R^a_{bjk} &= e_k L^a_{bj} - e_j L^a_{bk} + L^c_{bj} L^a_{ck} - L^c_{bk} L^a_{cj} - C^a_{bc} \Omega^c_{kj}, \\ R^i_{jka} &= e_a L^i_{jk} - D_k C^i_{ja} + C^i_{jb} T^b_{ka}, \\ R^c_{bka} &= e_a L^c_{bk} - D_k C^c_{ba} + C^c_{bd} T^c_{ka}, \\ R^i_{jbc} &= e_c C^i_{jb} - e_b C^i_{jc} + C^h_{jb} C^i_{hc} - C^h_{jc} C^i_{hb}, \\ R^a_{bcd} &= e_d C^a_{bc} - e_c C^a_{bd} + C^e_{be} C^a_{ed} - C^e_{bd} C^a_{ec}. \end{aligned} \quad (7)$$

Contracting the components of (7) we prove:

Corollary 6 a) *The Ricci d-tensor $\mathbf{R}_{\alpha\beta} \doteq \mathbf{R}^\tau_{\alpha\beta\tau}$ has the irreducible h- v- components*

$$R_{ij} \doteq R^k_{ijk}, \quad R_{ia} \doteq -R^k_{ika}, \quad R_{ai} \doteq R^b_{aib}, \quad R_{ab} \doteq R^c_{abc}. \quad (8)$$

b) *The scalar curvature of a d-connection is*

$$\overleftarrow{\mathbf{R}} \doteq \mathbf{g}^{\alpha\beta} \mathbf{R}_{\alpha\beta} = g^{ij} R_{ij} + h^{ab} R_{ab}.$$

c) *The Einstein d-densor is computed $\mathbf{G}_{\alpha\beta} = \mathbf{R}_{\alpha\beta} - \frac{1}{2} \mathbf{g}_{\alpha\beta} \overleftarrow{\mathbf{R}}$.*

In modern gravity theories one considers more general linear connections generated by deformations of type $\Gamma_\beta^\alpha = \widehat{\Gamma}_\beta^\alpha + \mathbf{P}_\beta^\alpha$. We can split all geometric objects into canonical and post-canonical pieces which results in N-adapted geometric constructions. For instance,

$$\mathcal{R}_\beta^\alpha = \widehat{\mathcal{R}}_\beta^\alpha + \mathbf{D}\mathcal{P}_\beta^\alpha + \mathcal{P}_\gamma^\alpha \wedge \mathcal{P}_\beta^\gamma \quad (9)$$

for $\mathcal{P}_\beta^\alpha = \mathbf{P}_{\beta\gamma}^\alpha \vartheta^\gamma$.

3 Anholonomic Frames and Nonmetricity in String Gravity

For simplicity, we investigate here a class of spacetimes when the nonmetricity and torsion have nontrivial components of type

$$\mathcal{T} \doteq \mathbf{e}_\alpha \rfloor \mathcal{T}^\alpha = \kappa_0 \phi, \quad \mathcal{Q} \doteq \frac{1}{4} \mathbf{g}^{\alpha\beta} \mathcal{Q}_{\alpha\beta} = \kappa_1 \phi, \quad \mathbf{\Lambda} \doteq \vartheta^\alpha \mathbf{e}^\beta \rfloor (\mathcal{Q}_{\alpha\beta} - \mathcal{Q} \mathbf{g}_{\alpha\beta}) = \kappa_2 \phi \quad (10)$$

where $\kappa_0, \kappa_1, \kappa_2 = const$ and $\phi = \phi_\alpha \vartheta^\alpha$. The abstract indices in (10) are "upped" and "lowed" by using $\eta_{\alpha\beta}$ and its inverse defined from the vielbein decompositions of d-metric, $\mathbf{g}_{\alpha\beta} = \mathbf{e}_\alpha^{\alpha'} \mathbf{e}_\beta^{\beta'} \eta_{\alpha'\beta'}$.

Let us consider the strengths $\mathbf{H}_{\nu\mu} \doteq \widehat{\mathbf{D}}_\nu\phi_\mu - \widehat{\mathbf{D}}_\mu\phi_\nu + W_{\mu\nu}^\gamma\phi_\gamma$ (intensity of ϕ_γ) and $\widehat{\mathbf{H}}_{\nu\lambda\rho} \doteq \mathbf{e}_\nu\mathbf{B}_{\lambda\rho} + \mathbf{e}_\rho\mathbf{B}_{\nu\lambda} + \mathbf{e}_\lambda\mathbf{B}_{\rho\nu}$ (antisymmetric torsion of the $\mathbf{B}_{\rho\nu} = -\mathbf{B}_{\nu\rho}$ from the bosonic model of string theory with dilaton field Φ) and introduce

$$\begin{aligned}\mathbf{H}_{\nu\lambda\rho} &\doteq \widehat{\mathbf{Z}}_{\nu\lambda\rho} + \widehat{\mathbf{H}}_{\nu\lambda\rho}, \\ \widehat{\mathbf{Z}}_{\nu\lambda} &\doteq \widehat{\mathbf{Z}}_{\nu\lambda\rho}\vartheta^\rho = \mathbf{e}_\lambda[\widehat{\mathbf{T}}_\nu - \mathbf{e}_\nu]\widehat{\mathbf{T}}_\lambda + \frac{1}{2}(\mathbf{e}_\nu[\mathbf{e}_\lambda]\widehat{\mathbf{T}}_\lambda)\vartheta^\gamma.\end{aligned}$$

We denote the energy-momentums of fields:

$$\Sigma_{\alpha\beta}^{[\phi]} \doteq \mathbf{H}_\alpha^\mu\mathbf{H}_{\beta\mu} - \frac{1}{4}\mathbf{g}_{\alpha\beta}\mathbf{H}^{\nu\mu}\mathbf{H}_{\nu\mu} + \mu^2(\phi_\alpha\phi_\beta - \frac{1}{2}\mathbf{g}_{\alpha\beta}\phi_\nu\phi^\nu),$$

$\mu^2 = \text{const}$, $\Sigma_{\alpha\beta}^{[mat]}$ is the source from any possible matter fields and $\Sigma_{\alpha\beta}^{[T]}(\widehat{\mathbf{T}}_\nu, \Phi)$ contains contributions of torsion and dilatonic fields.

Theorem 7 *The dynamics of sigma model of bosonic string gravity with generic off-diagonal metrics, effective matter and torsion and nonmetricity (10) is defined by the system of field equations*

$$\begin{aligned}\widehat{\mathbf{R}}_{\alpha\beta} - \frac{1}{2}\mathbf{g}_{\alpha\beta}\widehat{\mathbf{R}} &= k(\Sigma_{\alpha\beta}^{[\phi]} + \Sigma_{\alpha\beta}^{[mat]} + \Sigma_{\alpha\beta}^{[T]}), \\ \widehat{\mathbf{D}}_\nu\mathbf{H}^{\nu\mu} &= \mu^2\phi^\mu, \quad \widehat{\mathbf{D}}^\nu(\mathbf{H}_{\nu\lambda\rho}) = 0,\end{aligned}\tag{11}$$

where $k = \text{const}$, and the Euler-Lagrange equations for the matter fields are considered on background V_N^{n+m} .

Proof. See details in Ref. [2].

In terms of differential forms the eqs. (11) are written

$$\eta_{\alpha\beta\gamma} \wedge \widehat{\mathcal{R}}^{\beta\gamma} = \widehat{\Upsilon}_\alpha,\tag{12}$$

where, for the volume 4-form $\eta \doteq *1$ with the Hodge operator “*”, $\eta_\alpha \doteq \mathbf{e}_\alpha[\eta]$, $\eta_{\alpha\beta} \doteq \mathbf{e}_\beta[\eta_\alpha]$, $\eta_{\alpha\beta\gamma} \doteq \mathbf{e}_\gamma[\eta_{\alpha\beta}]$, ..., $\widehat{\mathcal{R}}^{\beta\gamma}$ is the curvature 2-form and Υ_α denote all possible sources defined by using the canonical d-connection. The deformation of connection (6) defines a deformation of the curvature tensor of type (9) but with respect to the curvature of the Levi-Civita connection, $\nabla\mathcal{R}^{\beta\gamma}$. The gravitational field equations (12) transforms into

$$\eta_{\alpha\beta\gamma} \wedge \nabla\mathcal{R}^{\beta\gamma} + \eta_{\alpha\beta\gamma} \wedge \nabla\mathcal{Z}^{\beta\gamma} = \widehat{\Upsilon}_\alpha,\tag{13}$$

where $\nabla\mathcal{Z}^\beta{}_\gamma = \nabla\mathcal{P}^\beta{}_\gamma + \mathcal{P}^\beta{}_\alpha \wedge \mathcal{P}^\alpha{}_\gamma$.

Corollary 8 *A subclass of solutions of the gravitational field equations for the canonical d-connection defines also solutions of the Einstein equations for the Levi-Civita connection if and only if $\eta_{\alpha\beta\gamma} \wedge \nabla\mathcal{Z}^{\beta\gamma} = 0$ and $\widehat{\Upsilon}_\alpha = \nabla\Upsilon_\alpha$, (i. e. the effective source is the same for both type of connections).*

Proof. It follows from the Theorem 7.

This property is very important for constructing exact solutions in Einstein and string gravity, parametrized by generic off-diagonal metrics and ahnolonomic frames with associated N-connection structure (see Refs. in [1, 2] and [3]) and equations (13)).

Let us consider a five dimensional ansatz for the metric (1) and frame (2) when $u^\alpha = (x^i, y^4 = v, y^5); i = 1, 2, 3$ and the coefficients

$$\begin{aligned} g_{ij} &= \text{diag}[g_1 = \pm 1, g_2(x^2, x^3), g_3(x^2, x^3)], h_{ab} = \text{diag}[h_4(x^k, v), h_5(x^k, v)], \\ N_i^4 &= w_i(x^k, v), N_i^5 = n_i(x^k, v) \end{aligned} \quad (14)$$

are some functions of necessary smooth class. The partial derivative are briefly denoted $a^\bullet = \partial a / \partial x^2, a' = \partial a / \partial x^3, a^* = \partial a / \partial v$.

4 Main results:

Theorem 9 *The nontrivial components of the Ricci d-tensors (8) for the canonical d-connection (6) are*

$$\begin{aligned} R_2^2 &= R_3^3 = -\frac{1}{2g_2g_3}[g_3^{\bullet\bullet} - \frac{g_2^\bullet g_3^\bullet}{2g_2} - \frac{(g_3^\bullet)^2}{2g_3} + g_2'' - \frac{g_2' g_3'}{2g_3} - \frac{(g_2')^2}{2g_2}], \\ R_4^4 &= R_5^5 = -\frac{1}{2h_4h_5}[h_5^{**} - h_5^*(\ln|\sqrt{|h_4h_5|}|)^*], \\ R_{4i} &= -w_i \frac{\beta}{2h_5} - \frac{\alpha_i}{2h_5}, \quad R_{5i} = -\frac{h_5}{2h_4}[n_i^{**} + \gamma n_i^*], \end{aligned} \quad (15)$$

$$\alpha_i = \partial_i h_5^* - h_5^* \partial_i \ln|\sqrt{|h_4h_5|}|, \quad \beta = h_5^{**} - h_5^* [\ln|\sqrt{|h_4h_5|}|]^*, \quad \gamma = 3h_5^*/2h_5 - h_4^*/h_4$$

$$h_4^* \neq 0 \text{ and } h_5^* \neq 0.$$

Proof. It is provided in Ref. [2].

Corollary 10 *The Einstein equations (12) for the ansatz (14) are compatible for vanishing sources and if and only if the nontrivial components of the source, with respect to the frames (3) and (2), are any functions of type*

$$\widehat{\Upsilon}_2^2 = \widehat{\Upsilon}_3^3 = \Upsilon_2(x^2, x^3, v), \quad \widehat{\Upsilon}_4^4 = \widehat{\Upsilon}_5^5 = \Upsilon_4(x^2, x^3) \text{ and } \widehat{\Upsilon}_1^1 = \Upsilon_2 + \Upsilon_4.$$

Proof. The proof, see details in [2], follows from the Theorem 9 with the nontrivial components of the Einstein d-tensor, $\widehat{\mathbf{G}}_\beta^\alpha = \widehat{\mathbf{R}}_\beta^\alpha - \frac{1}{2}\delta_\beta^\alpha \widehat{\mathbf{R}}$, computed to satisfy the conditions

$$G_1^1 = -(R_2^2 + R_4^4), G_2^2 = G_3^3 = -R_4^4(x^2, x^3, v), G_4^4 = G_5^5 = -R_2^2(x^2, x^3).$$

Having the values (15), we can prove [2] the

Theorem 11 *The system of gravitational field equations (11) (equivalently, of (12)) for the ansatz (14) can be solved in general form if there are given certain values of functions $g_2(x^2, x^3)$ (or, inversely, $g_3(x^2, x^3)$), $h_4(x^i, v)$ (or, inversely, $h_5(x^i, v)$) and of sources $\Upsilon_2(x^2, x^3, v)$ and $\Upsilon_4(x^2, x^3)$.*

Finally, we note that we have elaborated a new geometric method of constructing exact solutions in extra dimension gravity and general relativity theories. The classes of solutions define very general integral varieties of the vacuum and nonvacuum Einstein equations, in general, with torsion and nonmetricity and corrections from string theory, and/or noncommutative/quantum variables. For instance, in five dimensions, the metrics are generic off-diagonal and depend on four coordinates. So, we have proved in explicit form how it is possible to solve the Einstein equations on nonholonomic manifolds (see mathematical problems analized in Refs. [22, 23]), in our case, provided with nonlinear connection structure.

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